New faces of the Hermite-Hadamard inequality

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Left-hand side inequality

If $\Delta \subset \mathbb{R}^n$ is a simplex with vertices $\mathbf{x}_0, \mathbf{x}_1, \dots, \mathbf{x}_n$ and $f\colon \Delta \to \mathbb{R}$ is convex, then

$$f(\mathbf{b}) \le \operatorname{Avg}(f, \Delta),$$

where

$$\mathbf{b} = \frac{\mathbf{x}_0 + \mathbf{x}_1 + \cdots + \mathbf{x}_n}{n+1}$$

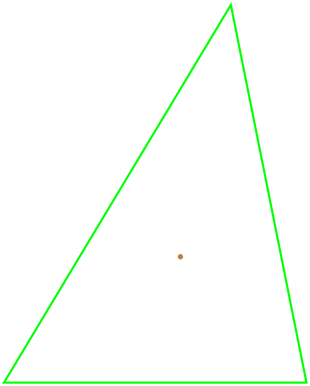
is the barycenter of Δ and

$$\operatorname{Avg}(f, \Delta) = \frac{1}{\operatorname{Vol} \Delta} \int_\Delta f(\mathbf{x}) d\mathbf{x}.$$

How can one improve LHS

$$\mathtt{f}(\mathbf{b}) \leq ? \leq \mathtt{Avg}(\mathtt{f}, \Delta)$$

Result of Dragomir and Raïssouli



$\mathcal{D}_0$ $\qquad$ $\mathrm{D}_0 = \mathrm{f}(\mathbf{b})$

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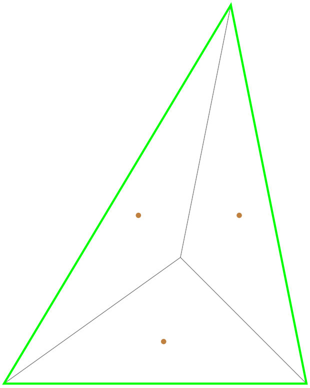

$\mathcal{D}_0 \qquad D_0 = f(\mathbf{b})$

$\mathcal{D}_1 \qquad D_1 = \frac{1}{|\mathcal{D}_1|} \sum_{\sigma \in \mathcal{D}_1} f(\mathbf{b}_\sigma)$

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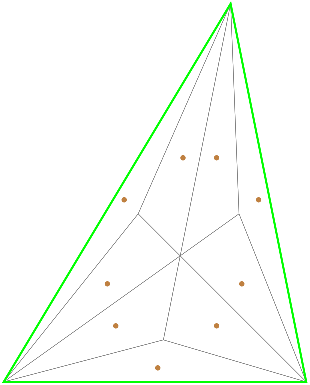

$\mathcal{D}_0$	$D_0 = f(\mathbf{b})$
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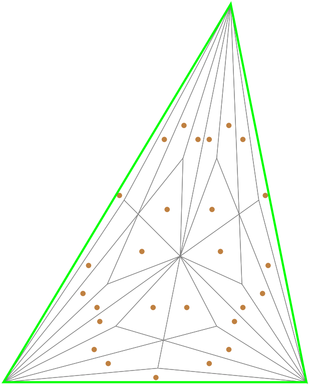

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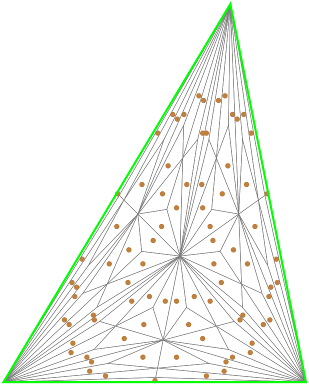

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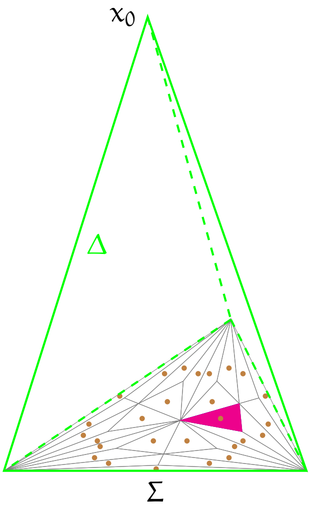
$$D_0 \le D_1 \le \cdots \le D_p \le D_{p+1} \to \mathtt{Avg}(f, \Delta)$$

Refinement of the left-hand side



$$\mathcal{D}_p \ni \sigma \longmapsto \delta = \mathrm{conv}(\sigma, x_0)$$

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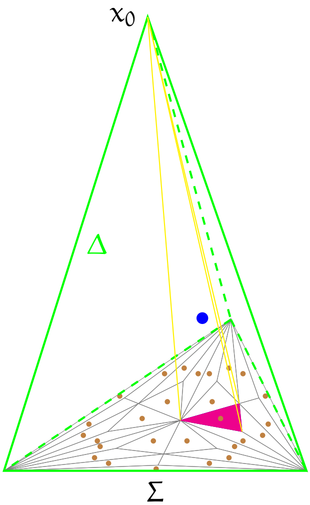

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$$f(\mathbf{b}_\delta) \le \frac{1}{\mathrm{Vol}\,\delta} \int_\delta f(\mathbf{x})d\mathbf{x}$$

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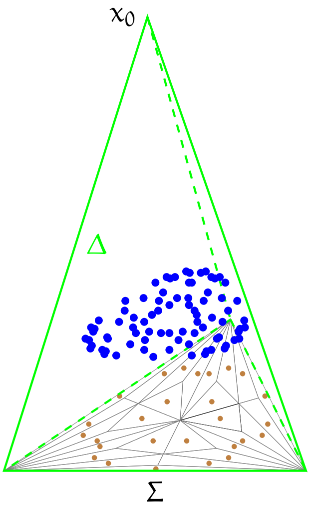

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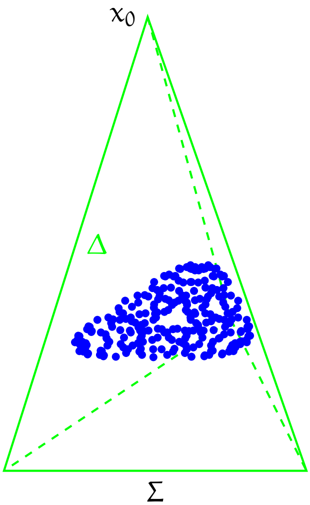

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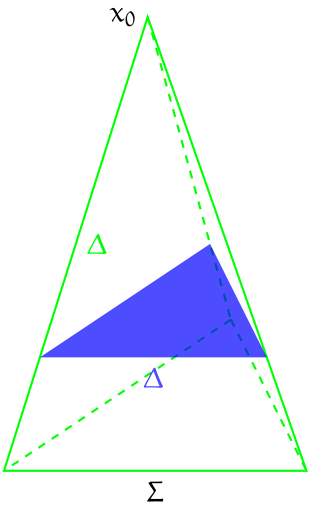

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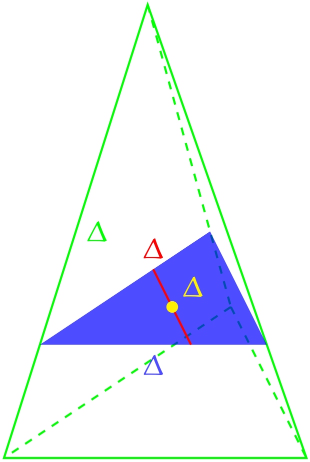

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Refinement of the left-hand side

For any subset K of N of cardinality $1 \le k \le n$ we define a $(k-1)$-simplex Δ^K as follows.

$$\Delta^K = H\left(b_{N\setminus K}, \frac{k}{n+1}\right)(\Delta_K) \tag{1}$$

where $\Delta_K = \mathrm{conv}\{x_i \colon i \in K\}$, and $H(a,\lambda)(x) = a + \lambda(x-a)$.

Refinement of the left-hand side

Theorem

If $f\colon \operatorname{conv}\{\mathbf{x}_0, \ldots, \mathbf{x}_n\} \to \mathbb{R}$ *is a convex function and* $\emptyset \neq K \subset L \subsetneq \{0, \ldots, n\}$, *then*

$$\operatorname{Avg}(f, \Delta^K) \leq \operatorname{Avg}(f, \Delta^L).$$

Refinement of the left-hand side

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Corollary

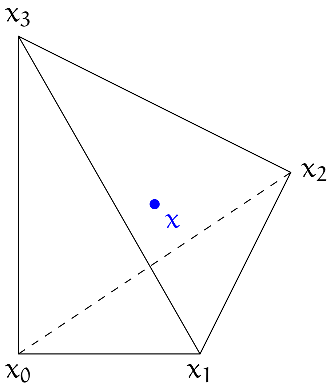
If $1 \le k < l \le n$, *then*

$$\frac{1}{\binom{n+1}{k}}\sum_{|K|=k}\operatorname{Avg}(f,\Delta^K) \le \frac{1}{\binom{n+1}{l}}\sum_{|L|=l}\operatorname{Avg}(f,\Delta^L).$$

How can one improve RHS

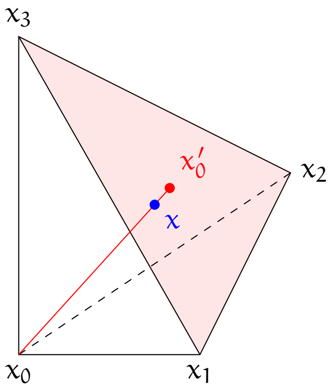
$$\mathrm{Avg}(f,\Delta) \le\ ?\ \le \frac{f(x_0)+\cdots+f(x_n)}{n+1}$$

Refinement of the right-hand side



$$x = \sum_{i=0}^{n} \alpha_i x_i$$

Refinement of the right-hand side

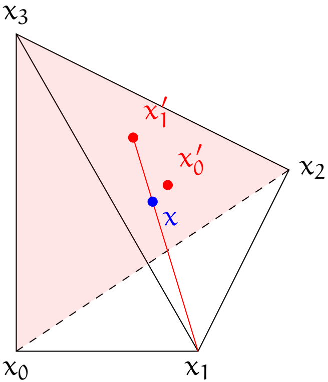


$$x = \sum_{i=0}^{n} \alpha_i x_i$$

$$x = \alpha_0 x_0 + \alpha_0' x_0'$$

$$(\alpha_i' = 1 - \alpha_i)$$

Refinement of the right-hand side



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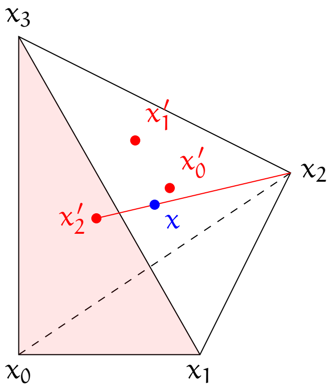

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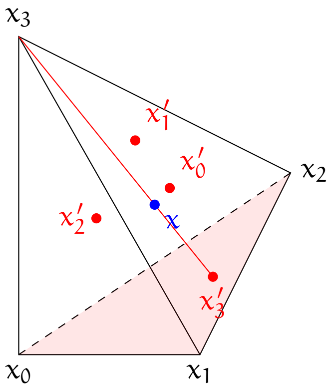

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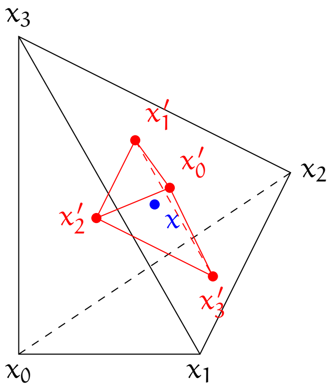

$$x = \sum_{i=0}^{n} \alpha_i x_i$$

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$$\dots$$

$$x = \alpha_n x_n + \alpha_n' x_n'$$

$$x = \sum_{i=0}^{n} \frac{\alpha_i'}{n} x_i' \Rightarrow f(x) \le \sum_{i=0}^{n} \frac{\alpha_i'}{n} f(x_i')$$

Refinement of the right-hand side

Integrating

$$f(\alpha_0 x_0 + \cdots + \alpha_n x_n) \le \sum_{i=0}^{n} \frac{\alpha_i'}{n} f(x_i')$$

over standard simplex one gets

$$\mathrm{Avg}(f, \Delta) \le \frac{1}{n+1} \sum_{\substack{K \subset N \\ |K|=n}} \mathrm{Avg}(f, \Delta_K)$$

New face of H-H

Let

$$L(k) = \frac{1}{\binom{n+1}{k}} \sum_{\substack{K \subset N \\ |K|=k}} \mathrm{Avg}(f, \Delta^K) \quad U(k) = \frac{1}{\binom{n+1}{k}} \sum_{\substack{K \subset N \\ |K|=k}} \mathrm{Avg}(f, \Delta_K)$$

Then

$$f\left(\frac{x_0 + \cdots + x_n}{n+1}\right) = L(1) \le L(2) \le \cdots \le L(n) \le \frac{1}{\mathrm{Vol}(\Delta)} \int_\Delta f(x)dx$$

$$\frac{1}{\mathrm{Vol}(\Delta)} \int_\Delta f(x)dx \le U(n) \le \cdots \le U(2) \le U(1) = \frac{f(x_0) + \cdots + f(x_n)}{n+1}$$

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Thank you

Details I

$$\int_{E_n} \alpha_0' f(x_0')\, d\alpha =$$
$$\int_{E_n} (\alpha_1 + \cdots + \alpha_n) f\left(\frac{\alpha_1}{\alpha_1 + \cdots + \alpha_n} x_1 + \cdots + \frac{\alpha_n}{\alpha_1 + \cdots + \alpha_n} x_n \right) d\alpha.$$

Let us perform the following change of variables:

$$t = \alpha_1 + \cdots + \alpha_n,\ \beta_2 = \alpha_2/t, \ldots, \beta_n = \alpha_n/t.$$

Obviously $0 \le t \le 1$ and $(\beta_2, \ldots, \beta_n) \in E_{n-1}$. This yields

$$\alpha_1 = t(1 - \beta_2 - \cdots - \beta_n),\ \alpha_2 = t\beta_2, \ldots, \alpha_n = t\beta_n$$

Details II

and the Jacobi determinant equals

$$\begin{vmatrix} 1-\beta_2-\cdots-\beta_n & \beta_2 & \beta_3 & \cdots & \beta_n \\ -t & t & 0 & \cdots & 0 \\ -t & 0 & t & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ -t & 0 & 0 & \cdots & t \end{vmatrix} = \begin{vmatrix} 1 & \beta_2 & \beta_3 & \cdots & \beta_n \\ 0 & t & 0 & \cdots & 0 \\ 0 & 0 & t & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & t \end{vmatrix} = t^{n-1}.$$

so

$$\int_0^1 t^n dt \int_{E_{n-1}} f((1-\beta_2-\cdots-\beta_n)x_1+\beta_2x_2+\ldots\beta_nx_n)d\beta$$
$$= \frac{1}{n+1}\frac{1}{(n-1)!\,\mathrm{Vol}\,\Delta_0}\int_{\Delta_0} f(x)dx.$$